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The force exerted on a body in inviscid unsteady non-uniform rotational flow

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A general expression is derived for the fluid force on a body of simple shape moving with a velocity $\boldsymbol{v}$ through inviscid fluid in which there is an unsteady non-uniform rotational velocity field $\boldsymbol{u}_0(\boldsymbol{x},t)$ in two or three dimensions. It is assumed that the radius is small compared with the scale over which the strain rate changes, though for the sphere it is also assumed that the changes in the ambient velocity field over the scale of the sphere are small compared with the velocity of the body relative to the flow. Given these approximations it is shown that the effects of the rate of change of the vorticity of the ambient flow is of second order and can be neglected. However the rate of change of the irrotational straining motion is included in the analysis. It is shown that the inertial forces derived by many authors for irrotational flow can be simply added to a generalization of the lift force derived by Auton (1987) in a companion paper. It is shown how this lift force is made up of a rotational and an inertial or added-mass component. For three-dimensional bluff bodies the latter is generally larger (by a factor of three for a sphere), and can be simply calculated from the added-mass coefficient. For illustration, the general expression is used to derive formulae for (i) the motion of a spherical bubble in a steady non-uniform flow to contrast with the motion in an unsteady flow, and (ii) the motion of rigid volumes of neutral density across an inviscid shear flow. These results show how added-mass (and lift) forces lead to different motions for a sphere and a cylinder. The general expression is useful in two-phase flow calculations, and for indicating the forces and motions of 'lumps of fluid' in turbulent flows.

1. Introduction

Students of fluid mechanics are generally familiar with the results for the force $\boldsymbol{F}$ on a rigid cylinder or sphere with volume $\mathscr{V}$ moving in an inviscid fluid with density ρ with a velocity $\boldsymbol{v}$, while the fluid has a spatially uniform ambient velocity $\boldsymbol{u}_0(t)$:

$$\boldsymbol{F} = \rho\mathscr{V}\left\{(1+C_M)\frac{\mathrm{d}\boldsymbol{u}_0}{\mathrm{d}t} - C_M\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right\}, \tag{1.1}$$

where C_M is the added-mass coefficient (Batchelor 1967). ($\mathscr{V}$ is the area for two-dimensional flow.)

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However, in rotational flows there are different forces. For example, a stationary circular cylinder in a steady shear flow $(-U-\omega_0 y, 0)$ experiences a lift force

$$F_y = \rho \mathscr{V} C_L U \omega_0, \tag{1.2a}$$

where $C_L = 2$. But if the cylinder is placed in a swirling flow with the same vorticity $(-U-\frac{1}{2}\omega_0 y, \frac{1}{2}\omega_0 x)$, the force is

$$F_y = \rho \mathscr{V} U \omega_0. \tag{1.2b}$$

Note the factor is 1 (Batchelor 1967, pp. 539–543). No general formula has been given for the case where the cylinder moves with velocity $\boldsymbol{v}$.

There are many basic and practical problems in fluid mechanics where it is desirable to have these formulae in a generalized form to allow for spatial and temporal variations of $\boldsymbol{u}_0$. The generalizations are only likely to be of the form of (1.1) or (1.2) if the diameter of the body is small compared with the scale over which there are changes in the gradients of the ambient (or undisturbed) fluid velocity.

Many analyses and suggestions have been made for the form of such 'generalized' force laws. Regrettably very many have been wrong, particularly in text books on two-phase flows. These errors and controversy have centred on the definition of the correct acceleration for the fluid. How should $\mathrm{d}\boldsymbol{u}_0/\mathrm{d}t$ be generalized to allow for the spatial variations in u_0 and the body's motion? Three suggestions have been made in the literature involving the undisturbed fluid velocity $\boldsymbol{U}(t) = \boldsymbol{u}_0(\boldsymbol{R}, t)$ at the body's location $\boldsymbol{R}(t)$ (see figure 1):

$$\text{(i)} \quad \left(\frac{\partial \boldsymbol{u}_0}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{u}_0(\boldsymbol{x}, t)\right)_{\boldsymbol{x}=\boldsymbol{R}} = \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t}, \tag{1.3}$$

the change of $\boldsymbol{u}_0$ seen by the 'body', (d/dt denotes the time derivative in a frame moving with the body);

or

$$\text{(ii)} \quad \left(\frac{\partial \boldsymbol{u}_0}{\partial t} + (\boldsymbol{v}-\boldsymbol{u})\cdot\nabla\,\boldsymbol{u}_0(\boldsymbol{x}, t)\right)_{\boldsymbol{x}=\boldsymbol{R}}, \tag{1.4}$$

no obvious interpretation:

or

$$\text{(iii)} \quad \left(\frac{\partial \boldsymbol{u}_0}{\partial t} + (\boldsymbol{u}_0\cdot\nabla)\,\boldsymbol{u}_0(\boldsymbol{x}, t)\right)_{\boldsymbol{x}=\boldsymbol{R}} = \left.\frac{\mathrm{D}\boldsymbol{u}_0}{\mathrm{D}t}\right|_{\boldsymbol{x}=\boldsymbol{R}} = \frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t}, \tag{1.5}$$

the change of $\boldsymbol{u}_0$ 'seen' by a fluid element in the absence of the body (where D/Dt denotes a derivative following a fluid element). (This controversy was reviewed by Thomas *et al.* 1983.)

The next question is whether the generalization of $\mathrm{d}\boldsymbol{u}_0/\mathrm{d}t$ is affected by the vorticity of the undisturbed flow, and whether the lift force can be suitably generalized. These are the questions that we shall answer in this paper. First we review what has been done already.

Taylor (1928) and Tollmien (1938) showed that the classical analysis for the rate of change of the kinetic energy of the flow around bodies in irrotational, non-uniform flows could be used to calculate the forces on the bodies. Taylor (1928) calculated the inviscid flow around a sphere in a steady irrotational non-uniform flow $(u_{01}(x_1), -x_2\,\partial u_{01}/\partial x_1, 0)$ and showed that the body (on the plane $x_2 = 0$) experiences a force $\rho(1+C_M)\mathscr{V} u_1\,\partial u_1/\partial x_1$.

The investigations of Taylor (1928) and Tollmien (1938) also showed this to be correct for a body of arbitrary shape when the relative direction of the flow is parallel to any one of its axes of permanent translation, so that it would experience no

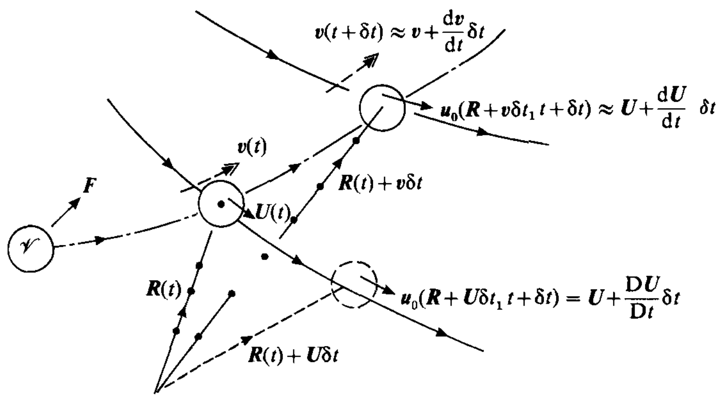

FIGURE 1. A rigid body with volume $\mathscr{V}$ at $R(t)$ moving with velocity $\boldsymbol{v}(t)$ through a non-uniform velocity field $\boldsymbol{u}(\boldsymbol{x},t)$. →—, streamlines; ·—·→—, position vector from the origin; ——→, body's track; ---≫, body's velocity (its acceleration is $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$); ——→——, velocity of the undisturbed flow, $\boldsymbol{u}_0$. At the particle $\boldsymbol{u}_0 = U(t)$. (Its rate of change, seen by the particle is $\mathrm{d}U/\mathrm{d}t$); ◌, the fluid volume at time $t+\delta t$, which coincided with the body at time t. (Its acceleration is $\mathrm{D}U/\mathrm{D}t$.)

couples in a uniform stream. If the body possesses point symmetry, and if straining motion of the flow is uniform, the forces depend only on the virtual-mass tensor† for that body. Taylor performed some ingenious wind-tunnel experiments to confirm that the inviscid theory gave the correct equilibrium positions of non-symmetric bodies in converging, diverging and curved flows.

Their analysis for irrotational motion, essentially rediscovered and generalized by Voinov, Voinov & Petrov (1973), Landweber & Miloh (1980), L'huillier (1982) and Auton (1983), yields the following expression for the force $\boldsymbol{F}$:

$$\boldsymbol{F} = -\mathscr{V}\left(\nabla P(1+C_M) + C_M \rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right), \tag{1.6}$$

where ρ is the density of the fluid, P is the pressure in the ambient flow, and C_M is the added-mass coefficient. $\boldsymbol{F}$ can also be written in terms of the acceleration in the ambient flow, at the location of the body's centre, $\mathrm{D}\boldsymbol{U}/\mathrm{D}t$, and in terms of $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$, the rate of change of the velocity of the body. Then we recover a form similar to that of (1.5), i.e.

$$\boldsymbol{F} = \rho\mathscr{V}\left((1+C_M)\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} - C_M\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right). \tag{1.7}$$

The contribution to the force by the term $-\mathscr{V}\nabla P$ in (1.6) is analogous to a 'buoyancy force' (Batchelor 1967).

Taylor's and Tollmien's results were overlooked by some later research workers studying forces on particles and bubbles, who assumed that the inertial forces on the body, induced by the velocity field $\boldsymbol{u}$, depended on the particle velocity $\boldsymbol{v}$ as well as $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$.

When a body moves in a rotational flow, the effects mentioned above may be present, but there will be a third force contributed by the lift effects. These lift forces are caused by the interaction between the vorticity and the relative velocity of the body with respect to the undisturbed flow.

† To first order in small quantities if the body is not point-symmetric, otherwise exactly.

Auton (1987) has calculated the lift force $\boldsymbol{F}_L$ on a sphere of volume $\mathscr{V}$ at rest in a simple shear flow, assuming a weak shear, with vorticity $\boldsymbol{\omega}$ which changes slowly with time. (These terms are defined more precisely in §3.) If the upstream velocity is $\boldsymbol{u}_0$,

$$\boldsymbol{F}_L = \rho C_L \mathscr{V} \boldsymbol{u}_0 \times \boldsymbol{\omega}. \tag{1.8}$$

The basis of Auton's calculations is Lighthill's (1956) 'Drift function method', which evaluates the vorticity change $\boldsymbol{\omega}_1$ due to the stretching of a uniform upstream vorticity $\boldsymbol{\omega}$ around an obstacle in a uniform flow.

The aim of this paper is to demonstrate that the expression for the lift force given by Auton has the same form when the flow is accelerating, and that the forces associated with the acceleration of the flow, hitherto only calculated for irrotational flow, can be added to the lift force. Auton (1983) suggested that such a generalization should be valid, without a detailed examination of its validity, and thence stated a general expression for the net force on a sphere:

$$\boldsymbol{F} = \rho \mathscr{V} \left\{ \left[(1+C_M)\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} - C_M \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \right] - C_L(\boldsymbol{v}-\boldsymbol{U}) \times \boldsymbol{\omega} \right\}, \tag{1.9}$$

where $\mathrm{D}\boldsymbol{U}/\mathrm{D}t$ and $\boldsymbol{U}$ are defined in (1.5) and in figure 1.

In §2 of the paper we derive the expression (1.9) exactly in the case of a cylinder accelerating in a two-dimensional, unsteady flow field as long as there is a linear variation of the unperturbed flow field, but these variations over the scale of the body do not need to be small. The vorticity distribution is constant and is therefore (in this case) known everywhere in the flow. The resultant force on the cylinder can be calculated exactly without assuming a weak shear flow. (The analysis of the flow field is quite standard, but a general formula for the force appears to be new.)

The problem of a sphere accelerating in a rotational straining flow is considered in §3. We show that the general expression (1.9) is still valid in this case in the limit of a weak slowly varying shear flow. The resultant force on the sphere can be calculated to first order in the spatial derivatives of the unperturbed flow. We show surprisingly that even in a shear flow the added-mass effects are important. Their effect on the force is greater than that of the vorticity distortion.

There are many applications of this apparently rather idealized flow. First, when large bluff bodies accelerate, the forces induced by the inviscid effects are often comparable with or larger than the drag forces. In fact Taylor's (1928) and Tollmien's (1938) studies were partly motivated by the need to estimate the forces on the unsteady motion of large airships in non-uniform flows. More recently, inviscid theory of flow round bodies has been applied to forces on marine structures in waves or currents, with or without motion (e.g. Sarpkaya & Isaacson 1981).

For smaller bodies this theory has been applied to bubbles, which, in sufficiently clean water and at high enough Reynolds number, can move as if the flow were inviscid. The general expression (1.9) has been used to model the motions of bubbles in the flow around a single horizontal vortex (Thomas *et al.* 1983) and of bubbles in the unsteady vortices of a vertical shear layer (Sene 1985; Hunt *et al.* 1988). Beyerlein, Cossmann & Richter (1985) and Auton (1983) have shown that the lift forces and the transverse pressure gradients play an important role in determining the distribution of bubbles in vertical turbulent pipe flows.

The general expression (1.9) is also useful in enabling us to calculate the forces on fluid volumes as they move across a shear flow. Prandtl (1925) supposed that lumps of fluid with finite volume move through a shear flow over a 'braking distance' and transfer momentum across the flow. But most discussions of mixing-length theory omit to mention how these forces are partly caused by the inertial force required to accelerate the 'added mass' around the 'lump of fluid' as it moves into a layer with greater velocity, and by the lift forces on the 'lump of fluid' as it moves across the shear flow. The example worked out in §4 illustrates the processes that are implicit in the general formula (1.5) and may also have some use in understanding the forces on eddies in turbulent flow (Hunt 1987). In particular the analysis shows the important difference between two- and three-dimensional 'lumps of fluid'.

2. Circular cylinder moving in a two-dimensional flow

2.1. *The velocity field*

Consider an infinite cylinder of radius a, located at $\boldsymbol{x} = \boldsymbol{R}(t)$, moving perpendicularly to its axis with a velocity $\boldsymbol{v}(t)$ in an inviscid and incompressible flow where the ambient flow field $\boldsymbol{u}_0(\boldsymbol{x}, t)$ is unsteady, rotational, non-uniform and two-dimensional in the (x_1, x_2)-plane normal to the cylinder axis (see figure 2). The uniform density of the fluid is ρ. We restrict ourselves to the case of an ambient flow with uniform vorticity and strain rate components. This is a good approximation for more general flows when the linear dimensions of the body are much smaller than the lengthscales of the flow variations.

The boundary-value problem to be solved is the calculation of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ around the cylinder and the force $\boldsymbol{F}$ per unit length acting on it, where $\boldsymbol{u}$ satisfies the Euler equation, written in terms of a moving or inertial coordinate system $\boldsymbol{x}$:

$$\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}\right) u_i \equiv \frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \tag{2.1a}$$

and the continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2.1b}$$

We shall also consider these equations expressed relative to a moving coordinate system $\boldsymbol{x}'$ with origin at the cylinder's centroid.

The boundary conditions on $\boldsymbol{u}$ specified in terms of the $\boldsymbol{x}$ and $\boldsymbol{x}'$ coordinates are

(i) no fluid passes through the cylinder, whence

$$\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{v} \cdot \boldsymbol{n} \quad \text{on} \quad |\boldsymbol{x}'| = |\boldsymbol{x} - \boldsymbol{R}| = a, \tag{2.2a}$$

where $\boldsymbol{n}$ is the outward normal, and $\boldsymbol{R}$ is defined as

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) + \int_0^t \boldsymbol{v}(\tau)\,\mathrm{d}\tau, \quad \boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{R},$$

assuming both coordinate systems coincide at time $t = 0$.

(ii) The flow tends to its undisturbed value far from the cylinder as

$$\frac{|\boldsymbol{x}'|}{a} \rightarrow \infty, \quad u_i \rightarrow u_{0i} = x_j e_{ij} + \tfrac{1}{2} e_{ijk}\,\omega_j x_k$$
$$= U_i(t) + x'_j e_{ij} + \tfrac{1}{2} e_{ijk}\,\omega_j x'_k, \tag{2.2b}$$

where $\boldsymbol{U}(t) = \boldsymbol{u}_0(\boldsymbol{R}(t), t)$. In a two-dimensional flow in the (x_1, x_2)-plane, the only non-zero components of e_{ij} and ω_j are e_{11}, $e_{22} (\equiv e_{11})$, e_{21}, $(e_{12} = e_{21})$ and ω_3. Note that e_{ij} and ω_j are uniform in space but may be functions of time.

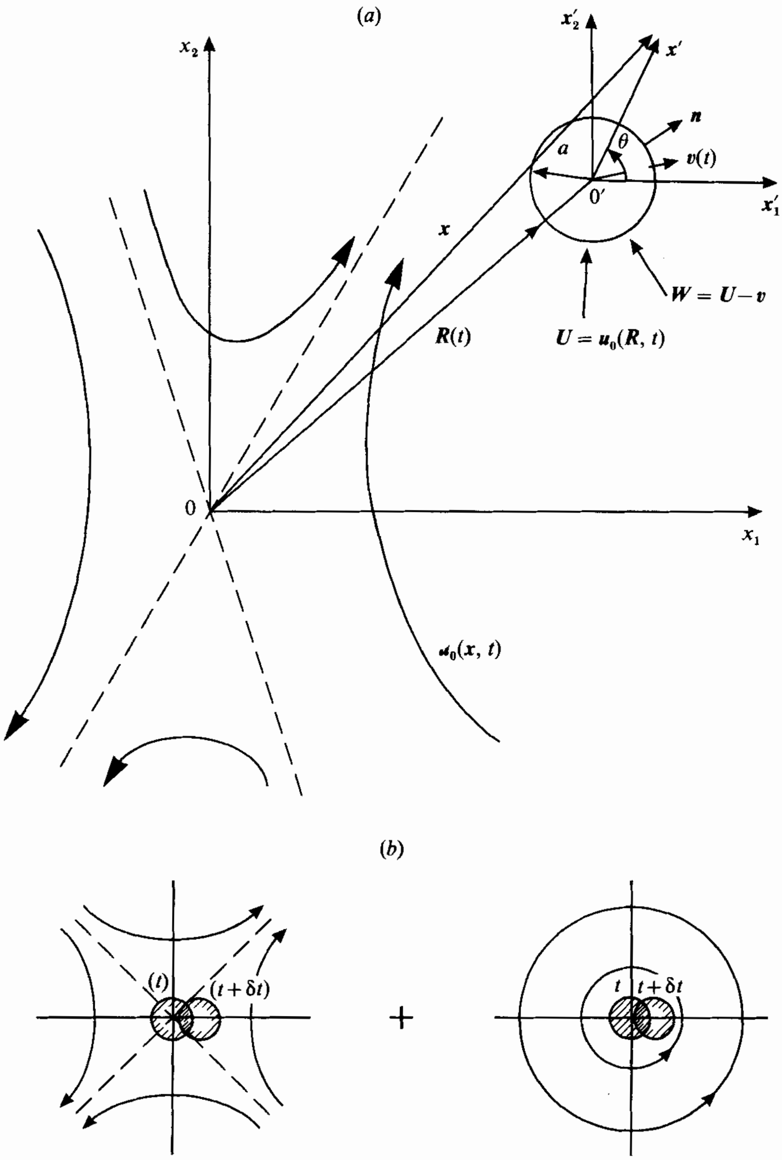

FIGURE 2. Cylinder moving in two-dimensional flow. (a) shows the straining flow given by (2.2b); (b) shows the irrotational and rotational components of the straining. It also shows a cylinder moving with a velocity $(v_1, 0)$ at 0. There is no force in the pure irrotational strain, but there is a vertical force on the cylinder in the rotational strain (equation (2.15b)).

From the curl of (2.1a) it follows that in a two-dimensional flow, the vorticity of each fluid element does not change, i.e.

$$\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}\right) \nabla \times \boldsymbol{u} = \frac{\mathrm{D}}{\mathrm{D}t} \nabla \times \boldsymbol{u} = 0.$$

So if ω_j is uniform in space (i.e. $\partial \omega_j / \partial x_i = 0$), it follows that

$$\frac{\partial \omega_j}{\partial t} = 0. \tag{2.3a}$$

Thus while the strain rate e_{ij} may vary on the timescale of relevance (a/ω), ω_j does not. Consequently, the vorticity in the flow around the cylinder is constant everywhere,

$$\nabla \times \boldsymbol{u} = \boldsymbol{\omega}. \tag{2.3b}$$

The solution to (2.3*b*) for $\boldsymbol{u}$ subject to the conditions (2.1*b*), (2.1*a*) and (2.2*b*) is a straightforward addition of a rotational velocity field (which is not affected by the cylinder) and an irrotational component, viz.

$$\boldsymbol{u} = \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{x}' + \nabla \phi, \tag{2.4a}$$

where ϕ must satisfy the following equations: from (2.2*a*),

$$\boldsymbol{n} \cdot \nabla \phi = \boldsymbol{v} \cdot \boldsymbol{n}; \tag{2.4b}$$

from (2.2*b*),

$$\frac{|\boldsymbol{x}'|}{a} \to \infty, \quad \phi = \tfrac{1}{2} x'_k x'_l e_{kl} + U_l x'_l; \tag{2.4c}$$

and from (2.1*b*)

$$\nabla^2 \phi = 0. \tag{2.4d}$$

For calculating the pressure field, it is convenient to express the solution in terms of the relative velocity field $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ seen by an observer at rest in the non-inertial frame $\boldsymbol{x}'$, moving with the body. We obtain as a solution to (2.4)

$$\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{v}, \tag{2.5a}$$

where

$$\boldsymbol{w} = \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{x}' + \nabla \phi', \tag{2.5b}$$

and

$$\phi' = \boldsymbol{W} \cdot \boldsymbol{x}' \left(1 + \frac{a^2}{r^2}\right) + \tfrac{1}{2} e_{12} \left(\frac{a^4}{r^2} + r^2\right) \sin 2\theta + \tfrac{1}{2} e_{11} \left(\frac{a^4}{r^2} + r^2\right) \cos 2\theta. \tag{2.5c}$$

Here $\boldsymbol{W}$ is the relative velocity between the ambient flow and the body, defined by

$$\boldsymbol{W} = \boldsymbol{U} - \boldsymbol{v}. \tag{2.5d}$$

2.2. *The surface pressure and net force*

In this non-inertial $\boldsymbol{x}'$ frame the pressure field $p'(\boldsymbol{x}, t)$ can be expressed in terms of $\boldsymbol{w}$ as

$$-\frac{1}{\rho} \nabla p' = \left(\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{w} \cdot \nabla) \boldsymbol{w}\right) + \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}. \tag{2.6a}$$

Note that since the surface of the cylinder is a streamline (in the $\boldsymbol{x}'$ coordinate, and since $\partial \boldsymbol{\omega}/\partial t = 0$ (from (2.3*a*)), the variation of pressure along the surface is given by

$$-\frac{1}{\rho} \nabla p' \times \boldsymbol{n} = \left(\nabla \left(\frac{\partial \phi'}{\partial t} + \tfrac{1}{2} \boldsymbol{w} \cdot \boldsymbol{w}\right) + \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right) \times \boldsymbol{n}. \tag{2.6b}$$

Thence the force per unit depth is

$$\boldsymbol{F} = \rho \int_S \left(\frac{\partial \phi'}{\partial t} + \tfrac{1}{2} \boldsymbol{w} \cdot \boldsymbol{w}\right) \boldsymbol{n} \, \mathrm{d}S + \rho \mathscr{V} \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}, \tag{2.7}$$

where the cross-sectional area πa^2, is expressed for greater generality as $\mathscr{V}$, the volume per unit depth.

Substituting (2.5) into (2.7) yields an expression for the force component per unit depth F_i on the cylinder in the inertial $\boldsymbol{x}_i$ frame:

$$F_i = \rho \mathscr{V}\{(1+C_M)(\dot{W}_i + e_{ij} W_j) + C_{L\Omega}\,\epsilon_{ijk} W_j \omega_k + \dot{v}_i\}, \tag{2.8}$$

where $C_M = 1$ is the virtual (or added)-mass coefficient and $C_{L\Omega} = 1$ is the rotational lift coefficient.

It is convenient to express (2.8) in terms of the material derivative of u_{0i} evaluated at the instantaneous position of the centroid, which we denote by $\mathrm{D}U_i/\mathrm{D}t$, using the standard relation between $\mathrm{D}U_i/\mathrm{D}t$ and the rate-of-strain tensor e_{ij} and the rate of rotation $\boldsymbol{w}$:

$$\frac{\mathrm{D}U_i}{\mathrm{D}t} = \left.\frac{\mathrm{D}u_{0i}}{\mathrm{D}t}\right|_{x_i=R_i} = \frac{\partial u_{0i}}{\partial t} + u_{0j}\, e_{ij} - \tfrac{1}{2}\epsilon_{ijk}\, u_{0j}\, \omega_k = -\frac{1}{\rho}\frac{\partial P_0}{\partial x_i} \quad (\boldsymbol{x} = \boldsymbol{R}), \tag{2.9}$$

where P_0 is the pressure in the ambient flow. Note that this differs from the rate of change of the undisturbed fluid velocity $\mathrm{d}\boldsymbol{U}/\mathrm{d}t$ seen by an observer travelling with the body, defined by

$$\frac{\mathrm{d}U_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} u_{0i}(\boldsymbol{R}(t), t) = \frac{\mathrm{d}R_j}{\mathrm{d}t}\frac{\partial u_{0i}}{\partial x_j} + \frac{\partial u_{0i}}{\partial t} = v_j \frac{\partial u_{0i}}{\partial x_j} + \frac{\partial u_{0i}}{\partial t}. \tag{2.10}$$

The expression (2.8) involving the relative velocity $\boldsymbol{W}$ and its rate of change can also be written in terms of e_{ij}, ω_k and $\mathrm{D}U_i/\mathrm{D}t$, using (2.5$d$) and (2.10), as

$$\dot{W}_i + e_{ij} W_j = \frac{\partial u_{0i}}{\partial t} + u_j\, e_{ij} - \tfrac{1}{2}\epsilon_{ijk}\, v_j\, \omega_k - \dot{v}_i, \tag{2.11}$$

$$= \frac{\mathrm{D}U_i}{\mathrm{D}t} + \tfrac{1}{2}\epsilon_{ijk}(U_j - v_j)\,\omega_k - \frac{\mathrm{d}v_i}{\mathrm{d}t}. \tag{2.12}$$

From (2.8), (2.11) and (2.12) therefore, the force can be expressed using vector notation solely in terms of the body's motion and the properties of the undisturbed velocity field, specifically the fluid and body's accelerations $\mathrm{D}\boldsymbol{U}/\mathrm{D}t$, $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$, and the fluid vorticity $\boldsymbol{w}$; viz.

$$\boldsymbol{F} = \rho \mathscr{V}\left\{(1+C_M)\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} + C_L(\boldsymbol{U}-\boldsymbol{v})\times\boldsymbol{\omega} - C_M\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right\}, \tag{2.13}$$

where $\mathrm{D}\boldsymbol{U}/\mathrm{D}t = -(1/\rho)\boldsymbol{\nabla} P$ is the ambient fluid acceleration at $\boldsymbol{x} = \boldsymbol{R}$, and $\boldsymbol{U} = \boldsymbol{u}_0(\boldsymbol{x} = \boldsymbol{R})$. Note that C_L is the conventional lift coefficient for shear flows which is related to the rotational lift coefficient $C_{L\Omega}$ by

$$C_L = \tfrac{1}{2}(1+C_M) + C_{L\Omega}. \tag{2.14}$$

This shows how the magnitude of C_L depends on the combined effects of the rotationality and inertially induced forces. For a circular cylinder $C_L = 2$ (Batchelor 1967, p. 543).

The result expressed by (2.13) is an exact one as long as the strain rates and vorticity of the undisturbed flow $\boldsymbol{u}_0$ are uniform as implied by (2.2b), without any restrictions on their magnitude. The lift force for an accelerating cylinder has the same form as when the cylinder is at rest or moving with uniform velocity $\boldsymbol{v}$. A buoyancy term $-(\rho\mathscr{V} - m)\boldsymbol{g}$ can be added on the right-hand side of (2.13) if needed.

2.3. *Example of a cylinder in a vortex*

Consider a cylindrical or spherical body at $\boldsymbol{x} = 0$ moving at a velocity $\boldsymbol{v}$ through a uniform rotational motion and an irrotational straining motion (i.e. $\boldsymbol{R} = 0$ in the flow defined by (2.2*b*)) (see figure 2*b*). At the centre of a forced vortex which is a purely rotational flow where $e_{ij} = 0$, the spatial gradients of velocity around the small body, in the frame of reference moving with the body, give rise to a rotational lift force $-\rho \mathscr{V} C_{L\Omega} \boldsymbol{v} \times \boldsymbol{\omega}$. This is the contribution of $\frac{1}{2}|\boldsymbol{\omega}|^2$ in the integrand in (2.7).

But in this frame the flow is unsteady, and far from the body the flow is accelerating perpendicular to the direction of motion of the body at a rate $-\frac{1}{2}\boldsymbol{v} \times \boldsymbol{\omega}$. This produces an 'inertial lift force'

$$-\tfrac{1}{2}\rho \mathscr{V}(1+C_M)\, \boldsymbol{v} \times \boldsymbol{\omega}.$$

Therefore the sum of these two lift forces, which can be expressed in terms of the lift coefficient given in (2.14) is

$$-\rho \mathscr{V} C_L\, \boldsymbol{v} \times \boldsymbol{\omega}. \tag{2.15}$$

So the usual lift coefficient C_L is really a sum of the rotational lift coefficient and a contribution by the added-mass coefficient.

Note that (2.15) agrees with (2.13) because at the centre of the vortex $\mathrm{D}\boldsymbol{u}_0/\mathrm{D}t = 0$.

In contrast, for a body moving through the centre of a linear irrotational straining field ($\omega_i = 0$), the force is zero because $\omega_i = 0$, as well as the acceleration being zero, i.e. $\mathrm{D}\boldsymbol{U}/\mathrm{D}t = 0$. The force is only non-zero when the particle is not at the centre, when $\boldsymbol{u}_0(\boldsymbol{R}, t) = (\boldsymbol{U}(t) \neq 0$.

In a simple shear flow, e.g. $\boldsymbol{u}_0 = (x_2\, \mathrm{d}u_1/\mathrm{d}x_2, 0, 0)$, $\mathrm{D}\boldsymbol{u}_0/\mathrm{D}t = 0$ everywhere. Only the lift force and the body's inertia terms ($\rho C_M \mathscr{V} \mathrm{d}\boldsymbol{v}/\mathrm{d}t$) are non-zero.

3. A sphere in an unsteady, three-dimensional straining flow

3.1. *The velocity field around the sphere*

Now consider a sphere of radius a, located at $\boldsymbol{x} = \boldsymbol{R}(t)$ moving with a velocity $\boldsymbol{v}(t)$ in an unsteady, inviscid, incompressible, uniform straining flow field $\boldsymbol{u}_0(\boldsymbol{x}, t)$ which is rotational and three-dimensional. Our aim is to use a calculation of $\boldsymbol{u}(\boldsymbol{x}, t)$ to derive a general expression for the force $\boldsymbol{F}(t)$ acting on the sphere.

The governing equations of the flow are given by (2.1) and the boundary conditions are also given by (2.2*a*, *b*). However, e_{ij} may have all nine components non-zero and ω_j have all three non-zero.

Because the flow is three-dimensional and rotational, the solution to (2.1) requires us to consider the full equation for the vorticity $\nabla \times \boldsymbol{u}$, viz.

$$\frac{\mathrm{D}}{\mathrm{D}t} \nabla \times \boldsymbol{u} = (\nabla \times \boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}. \tag{3.1}$$

Unlike the solution to (2.3*b*) for a two-dimensional body, two additional assumptions have to be made to solve the three-dimensional flow around the sphere. First the change of the velocity of the straining flow over the radius of the sphere, $a\|\nabla \boldsymbol{u}_0\|$, is small compared with the relative velocity $W = |\boldsymbol{W}|$ between the sphere and the fluid, i.e.

$$\epsilon = a\|\nabla \boldsymbol{u}_0\|/W \ll 1. \tag{3.2a}$$

This small parameter ϵ is useful as a measure of any errors involved in simplification of the analysis. Secondly we have to assume that the time for a change of $\boldsymbol{W}$ is large compared with the time it takes for a fluid element to pass around the volume, i.e.

$$\left|\frac{\partial \boldsymbol{W}}{\partial t}\right| \ll \frac{W^2}{a}. \tag{3.2b}$$

The upstream vorticity $\boldsymbol{\omega}$ is uniform in a uniform-straining flow field. But it may be unsteady. We can estimate the order of magnitude of the rate of change of $\boldsymbol{\omega}$ far from the body, from the vorticity equation, as

$$\left|\frac{\partial \boldsymbol{\omega}}{\partial t}\right| \approx |(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}_0| \approx \|\boldsymbol{\nabla}\boldsymbol{u}_0\|^2,$$

since $|\boldsymbol{\omega}| \approx \|\boldsymbol{\nabla}\boldsymbol{u}_0\|$. So in the time it takes a fluid element to go past the sphere ($\approx a/W$), the relative change in the far-field vorticity is given by

$$|\delta\boldsymbol{\omega}| \approx \|\boldsymbol{\nabla}\boldsymbol{u}_0\|^2\frac{a}{W} \ll |\boldsymbol{\omega}|. \tag{3.3}$$

Thus given the assumptions of (3.2) the time rate of change of $\boldsymbol{\omega}$ can be ignored in the ensuing analysis.

The solution to (3.1) is most easily obtained by considering the relative velocity field around the sphere,

$$\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}, \tag{3.4a}$$

and by expressing $\boldsymbol{w}$ as the far-field flow $\boldsymbol{w}_0(\boldsymbol{x}, t)$ and as the sum of components corresponding to perturbations to the different components of the far-field flow relative to the sphere, which are denoted by superscripts: for the uniform velocity$^{(\mathrm{U})}$, the extensional flow$^{(\mathrm{E})}$, the rotational flow$^{(\Omega)}$. Thus

$$\boldsymbol{w} = \Delta\boldsymbol{w}^{(\mathrm{U})} + \Delta\boldsymbol{w}^{(\mathrm{E})} + \Delta\boldsymbol{w}^{(\Omega)} + \boldsymbol{w}_0, \tag{3.4b}$$

where $\boldsymbol{w}_0 = \boldsymbol{u}_0 - \boldsymbol{v}$, and the first two components are irrotational and the second two are rotational. (This decomposit' m is possible because of the assumptions of (3.2).) Note that at the location of ' phere, $\boldsymbol{x} = \boldsymbol{R}, \boldsymbol{w}_0 = \boldsymbol{W}$.

Therefore the potential-now solution for $\Delta\boldsymbol{w}^{(\mathrm{U})}$ and $\Delta\boldsymbol{w}^{(\mathrm{E})}$ can be simply stated as

$$\Delta\boldsymbol{w}^{(\mathrm{U})} = +\boldsymbol{\nabla}\phi^{(\mathrm{U})}, \quad \Delta\boldsymbol{w}^{(\mathrm{E})} = +\boldsymbol{\nabla}\phi^{(\mathrm{E})}, \tag{3.5a}$$

where
$$\nabla^2\phi^{(\mathrm{U})} = \nabla^2\phi^{(\mathrm{E})} = 0, \tag{3.5b}$$

and
$$\left.\begin{aligned} \frac{\partial\phi^{(\mathrm{U})}}{\partial n} &= -\boldsymbol{w}_0\cdot\boldsymbol{n} = W_i n_i \\ \frac{\partial\phi^{(\mathrm{E})}}{\partial n} &= \qquad\quad\; = e_{ij}x'_j n_i \end{aligned}\right\} \quad \text{on} \quad |\boldsymbol{x}'| = a, \tag{3.5c}$$

where $\boldsymbol{W} = \boldsymbol{u}_0(\boldsymbol{R}, t) - \boldsymbol{v}$.

Note that the relative orders of magnitude of the components of $\boldsymbol{w}$ follow from (3.2), and are such that

$$\boldsymbol{w}_0 \approx \Delta\boldsymbol{w}^{(\mathrm{U})} = O(W), \quad \Delta\boldsymbol{w}^{(\mathrm{E})} \approx \Delta\boldsymbol{w}^{(\Omega)} = O(a\,\|\boldsymbol{\nabla}\boldsymbol{u}_0\|). \tag{3.6}$$

So the latter two components are smaller (by $O(\epsilon)$) then the former.

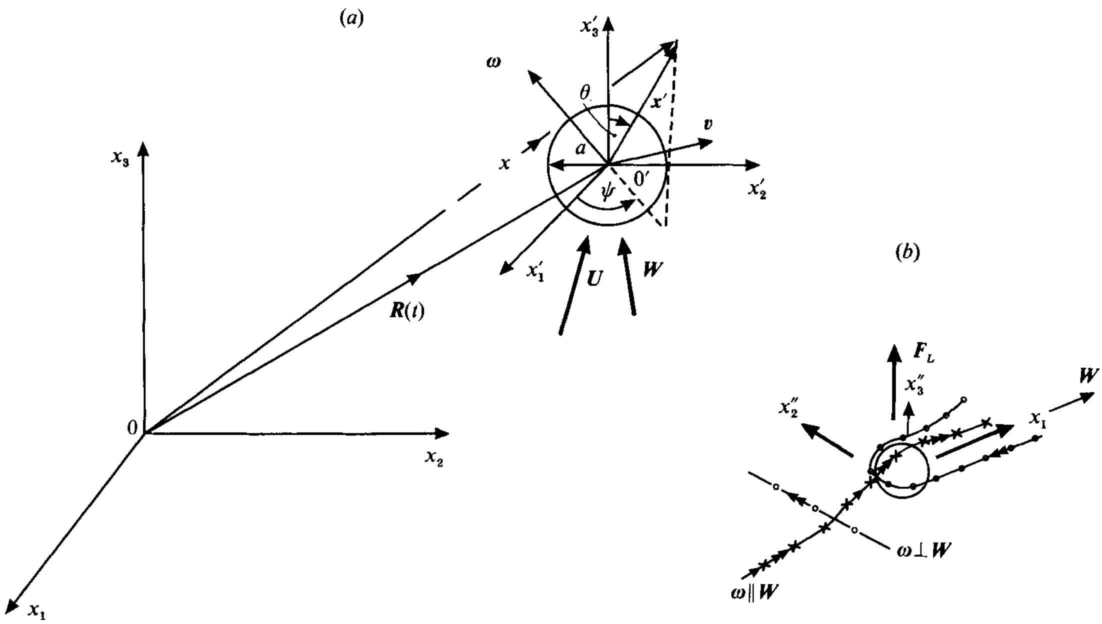

FIGURE 3. Sphere moving in a two-dimensional flow. (*a*) The coordinate system; (*b*) shows how the vorticity parallel and normal to the relative velocity W is distorted (when the rates of change of W and of the undisturbed vorticity ω are small on a timescale $a/|W|$).

The rotational perturbation $\Delta w^{(\Omega)}$ must satisfy the vorticity equation (3.1), which in the x' coordinate system is

$$\left(\frac{\partial}{\partial t}+(w\cdot\nabla)\right)(\nabla\times w) = (\nabla\times w)\cdot\nabla w. \tag{3.7}$$

This can be simplified, using the order-of-magnitude estimates in (3.3) and (3.6), to

$$((W+\Delta w^{(\mathrm{U})})\cdot\nabla)(\nabla\times w) = ((\nabla\times w)\cdot\nabla)(\Delta w^{(\mathrm{U})}) \tag{3.8}$$

where $\nabla\times w = \nabla\times w_0+\nabla\times\Delta w^{(\Omega)}$.

The errors are of order ϵ. But note that the change in the local vorticity $\nabla\times\Delta w^{(\Omega)}$ is the same order as the far-field vorticity $|\omega_0|$.

An important conclusion of (3.8) is that the distortion of vorticity $(\nabla\times\Delta w^{(\Omega)})$ around the sphere depends only on the form of the velocity field around the body; its does not depend on the magnitude of W (or e_{ij}) because $\Delta w^{(\mathrm{U})}$ is linearly dependent on W. Thus $\nabla\times w^{(\Omega)}$ is only a function of x' and ω. The calculations of $\Delta w^{(\Omega)}$ then follow in principle from the continuity equation and the condition that $\Delta w^{(\Omega)}\cdot n = 0$ on the sphere's surface.

To calculate $\Delta w^{(\Omega)}$ from (3.8) requires considering a coordinate system x'' centred in the sphere and aligned so that one component, $0x_1''$, is parallel to the relative velocity vector $W = u_0(R)-v$.

Let $\Delta w^{(\Omega)}$ be separated into components induced by the far-field vorticity perpendicular and parallel to W, viz.

$$\Delta w^{(\Omega)} = \Delta w_{\perp}^{(\Omega)}+\Delta w_{\|}^{(\Omega)}. \tag{3.9}$$

The vorticity equation (3.8) shows how the vortex line elements are distorted as they are advected round the sphere by the irrotational velocity field $\boldsymbol{W}+\Delta\boldsymbol{w}^{(\mathrm{U})}$ (figure 3*b*). Those vortex line elements that are initially perpendicular to $\boldsymbol{W}$ undergo continuous stretching as they pass over the sphere. So neither the vorticity field $\nabla\times\Delta\boldsymbol{w}_{\perp}^{(\Omega)}$ nor the induced velocity field $\Delta\boldsymbol{w}_{\perp}^{(\Omega)}$ are symmetrical about the centreplane $0x_2''x_3''$. The order of magnitude of $\Delta\boldsymbol{w}_{\perp}^{(\Omega)}$ is just proportional to the component of $\boldsymbol{\omega}$ perpendicular to $\boldsymbol{W}$ and to the radius of the sphere, i.e. $a|\boldsymbol{W}\times\boldsymbol{\omega}|/|\boldsymbol{W}|$. The method of calculating $\Delta\boldsymbol{w}_{\perp}^{(\Omega)}$ in terms of the Lighthill drift function is given by Auton (1987).

Note that the vorticity near the surface of the sphere reaches a singular value, but the induced velocity remains finite and of order $|\boldsymbol{\omega}|a$.

On the other hand, the vortex elements that are initially parallel to $\boldsymbol{W}$ are distorted symmetrically because they lie on the streamlines of the potential flow (figure 3*b*). The solution to (3.8) for the vorticity field induced by the component of $\boldsymbol{\omega}$ that is parallel to $\boldsymbol{W}$ is

$$\nabla\times\boldsymbol{w}_{\parallel}^{(\Omega)}=\frac{(\boldsymbol{\omega}\cdot\boldsymbol{W})}{|\boldsymbol{W}|^2}(\boldsymbol{W}+\Delta\boldsymbol{w}^{(\mathrm{U})}). \tag{3.10}$$

This induces a weak, swirling velocity field $\Delta\boldsymbol{w}_{\parallel}^{(\Omega)}$ which is symmetrical about the centreplane $(0x_2''x_3'')$, since $\boldsymbol{W}+\Delta\boldsymbol{w}^{(\mathrm{U})}$ is symmetrical.

Summarizing the results for the velocity field of the relative motion $\boldsymbol{w}$, its components are defined by (3.4), (3.5) and (3.9).

3.2. *Calculating the force*

The force on the sphere can be calculated from the integral given in (2.7), since in this case also $\partial\boldsymbol{\omega}/\partial t$ is zero to first order in ϵ. Therefore

$$\boldsymbol{F}=\rho\int_S\left(\frac{\partial\phi'}{\partial t}+\tfrac{1}{2}\boldsymbol{w}\cdot\boldsymbol{w}\right)\boldsymbol{n}\,\mathrm{d}s+\rho\mathscr{V}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}, \tag{3.11a}$$

where $\phi'=\phi^{(\mathrm{U})}+\phi^{(\mathrm{E})}+W_i x_i+e_{ij}x_i x_j$ and, to $O(\epsilon)$,

$$\boldsymbol{w}\cdot\boldsymbol{w}=(\boldsymbol{W}+\nabla\phi^{(\mathrm{U})})\cdot\{\boldsymbol{W}+\nabla\phi^{(\mathrm{U})}+2[\nabla\phi^{(\mathrm{E})}+\Delta\boldsymbol{w}^{(\Omega)}]\}. \tag{3.11b}$$

Because of the symmetry, it follows that the contributions to the surface integral of $\boldsymbol{w}\cdot\boldsymbol{w}$ come from the perturbations caused by the external flow $\nabla\phi^{(\mathrm{E})}$, i.e. $(1+C_M)e_{ij}W_j$, and by the vorticity component perpendicular to the flow $\Delta\boldsymbol{w}_{\perp}^{(\Omega)}$. Since $\Delta\boldsymbol{w}_{\perp}^{(\Omega)}$ is proportional to $a(\boldsymbol{W}\times\boldsymbol{\omega})/|\boldsymbol{W}|$ and since the latter force, induced by $\Delta\boldsymbol{w}_{\perp}^{(\Omega)}$, is in the direction $\boldsymbol{W}\times\boldsymbol{\omega}$, the force must be proportional to $\rho(\boldsymbol{W}\times\boldsymbol{\omega})a^3$.

The contribution to $\boldsymbol{F}$ by the extensional flow has been obtained by previous authors (as cited in §1). So for a sphere (3.11) reduces to

$$\boldsymbol{F}=\rho\mathscr{V}\{(1+C_M)(\dot{W}_i+e_{ij}W_j)+C_{L\Omega}\epsilon_{ijk}W_j\omega_k+\dot{v}_i\}, \tag{3.12}$$

where the rotational lift coefficient $C_{L\Omega}$, which has not yet been specified, is independent of $\boldsymbol{v}$ or $\boldsymbol{u}_0$. For a sphere $C_M=\frac{1}{2}$.

As in (2.13), $\boldsymbol{F}$ can be written in terms of the Eulerian velocity field determined in a fixed frame as

$$\boldsymbol{F}=\rho\mathscr{V}\left\{(1+C_M)\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t}+C_L(\boldsymbol{U}-\boldsymbol{v})\times\boldsymbol{\omega}-C_M\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right\}, \tag{3.13}$$

where the material derivative is evaluated at $\boldsymbol{x}=\boldsymbol{R}$ and C_L is the lift coefficient defined by (2.14).

Auton (1987) has analysed the lift on a sphere fixed in a weak shear flow, $\boldsymbol{u} = (u_1(x_2), 0, 0)$. In that case, $\mathrm{D}U/\mathrm{D}t = 0$ and $\boldsymbol{v} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t = 0$, so

$$F_2 = \rho \mathscr{V} C_L \mathrm{u}_{01} \frac{\partial u_{01}}{\partial x_2}. \tag{3.14}$$

His analysis (supported by computations) shows that $C_L = \frac{1}{2}$. Since $C_M = \frac{1}{2}$, this implies that $C_{L\Omega} = -\frac{1}{4}$, in contrast with the positive value $C_{L\Omega} = 1$ for a circular cylinder. Because the vorticity is distorted by the sphere, (and not by the cylinder), for a sphere moving with velocity v_1 at the centre of a forced vortex ($\omega_3 > 0$), the velocity increases over the upper half ($x_1 > 0$) for a cylinder and on the lower half ($x_2 < 0$) for a sphere. This causes the rotational lift force generated by the spatial changes in the velocity field,

$$(\tfrac{1}{2}\rho \textstyle\int (\boldsymbol{w}\cdot\boldsymbol{w})\,\boldsymbol{n} \propto C_{L\Omega}\, \boldsymbol{W}\times\boldsymbol{\omega}),$$

to be negative for a sphere and positive for a cylinder. The reason why the total lift force is positive is because the far-field velocity relative to the body is changing, i.e. $\mathrm{d}\boldsymbol{W}/\mathrm{d}t \neq 0$. In this case, as it moves into the higher velocity flow, it is displacing fluid with higher velocity in the x_2 direction and therefore experiences a lift force. For a sphere this effect of 'added mass' overcomes the rotational lift force. In fact, this inertial lift force is three times larger than the rotational lift force for a sphere. This suggests that for most three-dimensional bluff bodies in inviscid flow C_L is largely determined by the added-mass contribution $\frac{1}{2}(1+C_M)$.

The result expressed by (3.13) is valid for weakly sheared flows, where only first-order effects in e_{ij} and ω_i are considered. Unlike (2.15), it is an asymptotic and not an exact result even when the unperturbed flow $\boldsymbol{u}_0$ is the uniform strain field specified by (2.2*b*).

4. The movement of a fixed shape in simple non-uniform shear flows

The first example we consider is a spherical bubble with zero mass on the centreline of a steady converging flow defined by $\boldsymbol{u}_0 = (U_0 + \alpha x_1, -\alpha x_2, 0)$. Since the body has zero mass, no net force can be acting on it, i.e. $F = 0$. Since the flow is irrotational, (3.13) reduces to

$$0 = (1+C_M)\, u_{01} \frac{\partial u_{01}}{\partial x_1} - C_M \frac{\mathrm{d}v_1}{\mathrm{d}t}.$$

Since $\mathrm{d}v_1/\mathrm{d}t = v_1 \mathrm{d}v_1/\mathrm{d}x$ and $C_M = \frac{1}{2}$, the change in the velocity of the bubble from its value at x_0 is given by

$$v_1^2(x) - v_1^2(x_0) = 3(u_{01}^2(x) - u_{01}^2(x_0)). \tag{4.1}$$

Note how this result contrasts with that for a bubble in a uniform unsteady flow where

$$v_1(t) - v_1(t_0) = 3\,(u_{01}(t) - u_{01}(t_0)) \tag{4.2}$$

(Batchelor 1967, §6.8). The result (4.1) is a useful upper limit for the speeds of bubbles in real fluids accelerating in a converging pipe flow (Kowe *et al.* 1988). Comparing (4.1) and (4.2) shows that the bubbles' increase in speed for given change in fluid velocity in a steady converging flow is less by at most a factor of $\surd 3$ than for given change in a uniform unsteady flow. The result (4.1) is plotted in figure 4(*b* and it is compared with its asymptotic forms for small and large time.

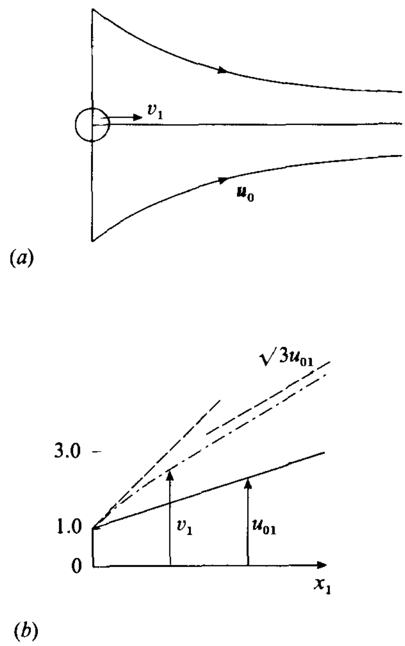

FIGURE 4. A spherical bubble accelerating in a converging flow. (a) the streamlines of the flow; (b) the velocity of the bubble v_1, and its asymptotic values compared with the liquid velocity u_{01}.

Consider now a volume of fixed shape with the same density as the fluid moving in the plane of a two-dimensional simple shear flow such as in figure 5. Let the flow $\boldsymbol{u}_0$ be parallel to x_1, so that $u_{01} = \alpha x_2$ and $\omega_3 = -\alpha$ are the only non-zero velocity and vorticity components for the unperturbed flow. The volume is given an initial velocity v_0 across the flow along x_2 at the origin at $t = 0$. If the volume is spherical, we assume that α is small enough for (3.13) to be accurate. Since $\mathrm{D}\boldsymbol{u}_0/\mathrm{D}t$ is zero throughout the flow, the equation for the velocity $\boldsymbol{v}$ of the volume (obtained at once from (2.13) or (3.13), is

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = B(\boldsymbol{U}-\boldsymbol{v}) \times \boldsymbol{\omega}, \tag{4.3}$$

with

$$B = \frac{C_L}{1+C_M}, \tag{4.4}$$

and the fluid velocity at the location of the volume is defined by

$$U = u_0(\boldsymbol{R}(t)) = \alpha R_2(t). \tag{4.5}$$

Solving (4.3) yields a simple expression for the streamwise velocity component v_1 of the volume

$$v_1 = BU_1 = B\alpha R_2,$$

so the difference between the velocity of the lump and of the fluid as it moves across the shear layer is

$$u_1' = (v_1 - U_1) = (B-1)\alpha R_2. \tag{4.6}$$

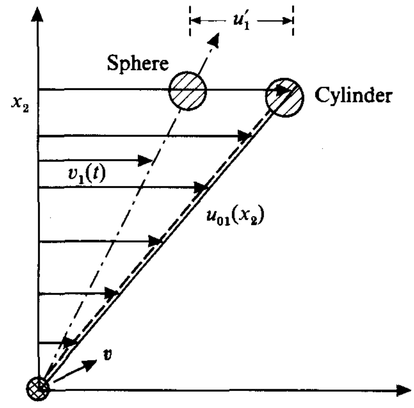

FIGURE 5. Rigid bodies with the same density as that of the fluid moving across a simple shear flow, $u_{01} = \alpha x_2$, showing how the mean flow acts to accelerate the bodies in the direction of the flow. ——, Shear-flow profile $u_{01}(x_2)$; ---, streamwise velocities of a sphere and cylinder $v_1(t)$ at $R_2(t) = x_2$. Note the difference u_1', between the ambient velocity and the velocity u_{01} and the velocity v_1 for a sphere. But as the cylinder moves across the shear flow, the streamwise forces are such that $u_1' = 0$.

This is the velocity perturbation u_1' associated with a fluid 'lump'. The cross-stream velocity is given by

$$v_2 = \begin{cases} v_0 \cosh\,[|\alpha|\,(B(1-B)\,t)^{\frac{1}{2}}], & B<1, \\ v_0 \quad , & B=1, \\ v_0 \cos\,[|\alpha|\,(B(1-B)\,t)^{\frac{1}{2}}], & B>1, \end{cases}$$

and

$$R_2(t) = \int_0^t v_2(t')\,\mathrm{d}t'.$$

These results differ qualitatively, depending on the value of B, owing solely to differences in the lift force.

Since for a cylinder $B = 1$, as it arrives in the faster fluid it accelerates to the same streamwise velocity by the action of the rotational and inertial forces acting on it, i.e. $u_1' = 0$. (If this was an eddy, there would be no effective stress acting on it.) But the lift force in the vertical direction is zero because $v_1 = U_1$.

But a sphere, for which $B = \frac{1}{3}$, does not accelerate so fast in the streamwise or x_1 direction because of its lower lift coefficient as it moves across the velocity gradient, so that its streamwise velocity is less than that of the fluid. Therefore $u_1' = -\frac{2}{3}t\,\partial u_{01}/\partial x_2$ for a small time after its release. (If this was an eddy, it would then experience a drag force exerted by the surrounding fluid, and effectively exert a Reynolds stress.) However this velocity defect increases the lift force tending to drive it faster across the flow (see figure 4). (Note that in thinking about particles and eddies it is often assumed that in a small deflection R_2 the change in v_1 is small, compared with $R_2\,\alpha$. We have seen here that this change can be calculated even when it is not small (Hunt 1987).)

Note that many authors have assumed that (3.12) has the form

$$\boldsymbol{F} = \rho\mathscr{V}\left[\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - C_M\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u})_{\boldsymbol{x}=\boldsymbol{R}(t)}\right)\right] \tag{4.7}$$

where
$$\frac{\mathrm{d}(\boldsymbol{u})_{x=R(t)}}{\mathrm{d}t} = \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right)_{x=R(t)},$$

is the change of fluid velocity following the position of the body. For a body moving across a shear flow, with no streamwise relative velocity, it is true that

$$\frac{\mathrm{d}(\boldsymbol{u})_{x=R}}{\mathrm{d}t} = -\boldsymbol{v}\times\boldsymbol{\omega}.$$

But to calculate the force on the body, the coefficient multiplying $(-\boldsymbol{v}\times\boldsymbol{\omega})$ must be C_L and not C_M, because there is a lift force as well as an added-mass force acting on the particle. For a sphere $C_L = C_M$, in which case (4.7) gives the correct force for its initial movements. But $C_L \neq C_M$ for a cylinder. In general, as we showed in (4.1), (4.7) is quite wrong.

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